Power patterns
Extending number sense through last digit investigations

John West
Edith Cowan University
<j.west@ecu.edu.au>

John West shows how using an investigative problem to employ problem solving and reasoning strategies can be used to investigate powers, place value and index notation to improve student understanding.

Introduction

Problems sourced from national and international competitions such as the Australian Mathematics Competition are often used effectively as enrichment and/or extension activities for high achieving students. While this has meant that they are sometimes deemed too esoteric or trivial for use in the regular classroom, this paper takes the view that the study of patterns and relationships is at the heart of mathematics and that such problems provide an excellent opportunity for students to develop a deeper understanding and appreciation of mathematical reasoning. Thus problems that may be dismissed as mere mathematical curiosities may in fact represent a valuable resource for the time-poor teacher.

Examination of past competition papers would suggest that mathematicians are preoccupied, for example, with problems like finding the last digit of numbers such as 3^{2015} (Atkins & Taylor, 2012). While students generally attempt to solve this type of problem using a brute-force approach (i.e., by calculating 3×3 then $3 \times 3 \times 3$ etc.), the limitations of such an approach are readily apparent. This is the case even with the assistance of a calculator, since the magnitude of the numbers involved means that the solution is too large to be accommodated on a calculator display. The need for an alternative problem-solving approach is clearly apparent, and a problem that on the surface may appear contrived and somewhat tedious (though not impossible) to solve, provides an ideal opportunity for students to explore the use of number patterns as a problem-solving technique.

Posamentier and Krulik (2012) suggest a range of strategies for motivating students in mathematics. These include: indicating a void in students' knowledge, discovering a pattern, presenting a challenge, enticing the class with a 'gee-whiz' mathematical result and getting students involved in justifying mathematical curiosities. The investigation described here incorporates several of these strategies as students discover the cyclical pattern that exists in the last digits of the powers of various digits. The investigation is intended as a possible activity for upper primary students, as Booker, Bond, Sparrow, and Swan (2014) caution "A concise notation using exponents can be introduced to record the product… However, because many children confuse the use of exponents with multiplication, their introduction should not occur until large numbers are familiar, multiplication itself is very secure, and children see the need for and power of this new way of recording numbers" (p. 146).

The relevant content description is taken from Year 7 of the *Australian Curriculum: Mathematics*, in which students "Investigate index notation and represent whole numbers as products of powers of prime numbers" (ACMNA149) and "Investigate and use square roots of perfect square numbers" (ACMNA150). This builds on earlier work from Year 4, in which students "Explore and describe number patterns resulting from performing multiplication" (ACMNA081), and Year 6 where

students “Identify and describe properties of prime, composite, square and triangular numbers” (ACMNA122) (Australian Curriculum and Reporting Authority, 2014). The problem solving and reasoning proficiency strands provide the contexts for developing these skills; according to Booker et al. (2014), “As much time and resources need to be allocated to establishing problem solving as to building competence with computational and other processes” (p. 14).

Method

It is suggested that the investigation described here is scaffolded using a SMART board or similar technology, as shown in Figure 1. Using the interactive ‘Powers’ manipulative (available from the SMART Notebook Gallery) allows the user to display integers raised to a wide range of integral powers. The teacher or student can use the arrows to increase or decrease the base and power to which it has been raised. As can be seen, the ‘Powers’ manipulative allows a wide range of index numbers to be displayed (up to 15 digits) before reverting to scientific notation.

The teacher should begin by reviewing several simple examples with the class to ensure they understand the process of calculating index numbers. Some time may need to be spent developing students’ understanding of index notation. Students must be able to clearly distinguish the base and the index, and must understand the difference between the index notation and multiplication e.g., 3^2 and 3×2. According to Haylock and Manning (2014), connecting the geometric idea of a square number as the number of items in a square array with the arithmetic idea that “a square number is any number that is obtained by multiplying (a whole) number by itself” (p. 274), helps to build understanding and confidence with number concepts. The process can be further scaffolded using Multibase Arithmetic Blocks (MABs) which provide a tangible illustration of what is meant by ‘squaring’ and ‘cubing’ a number in various bases (see Figure 2). Care must be taken, however, since the concept cannot be easily extended beyond cubes to explain the geometric significance of a^4. Given their geometric significance, it is desirable that students are familiar with the squares and cubes of small positive integers.

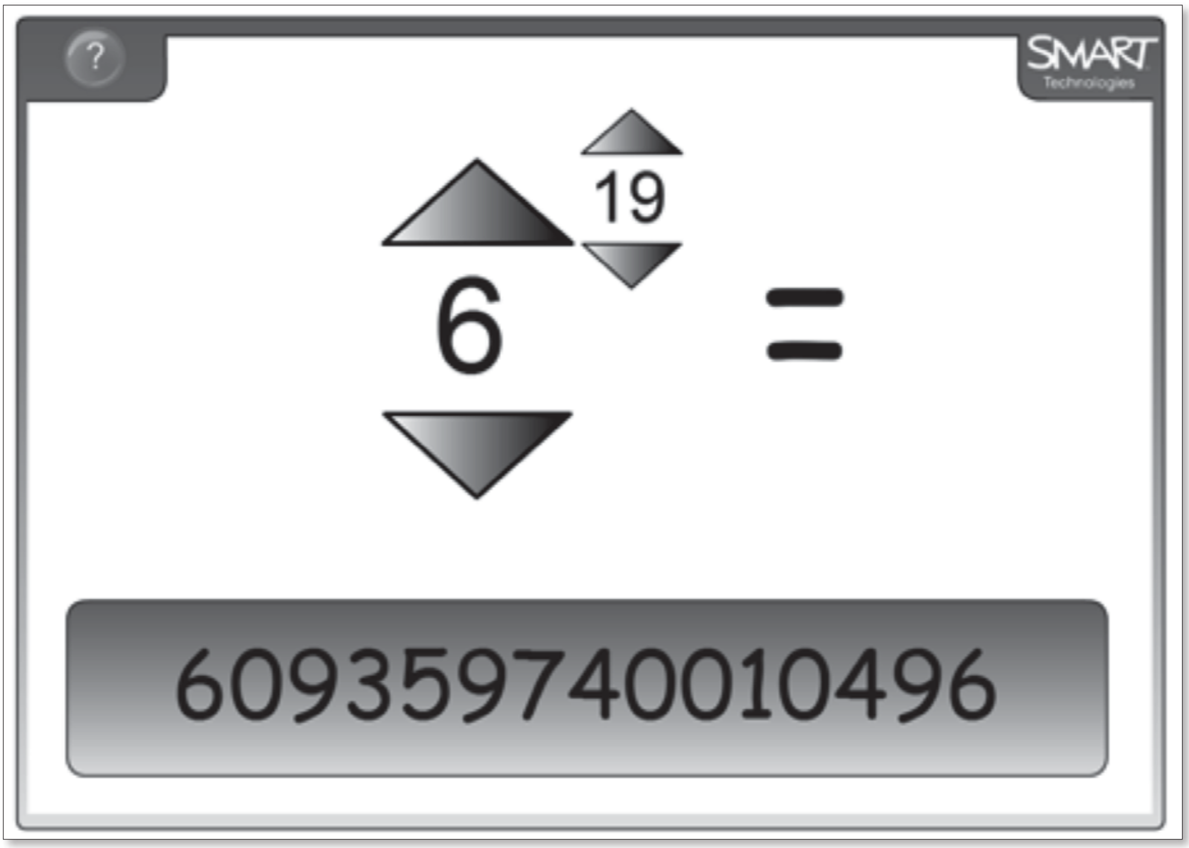

Figure 1. The *SMART 'Powers'* interactive gallery allows users to explore patterns in the last digits of numbers raised to a wide range of integral powers.

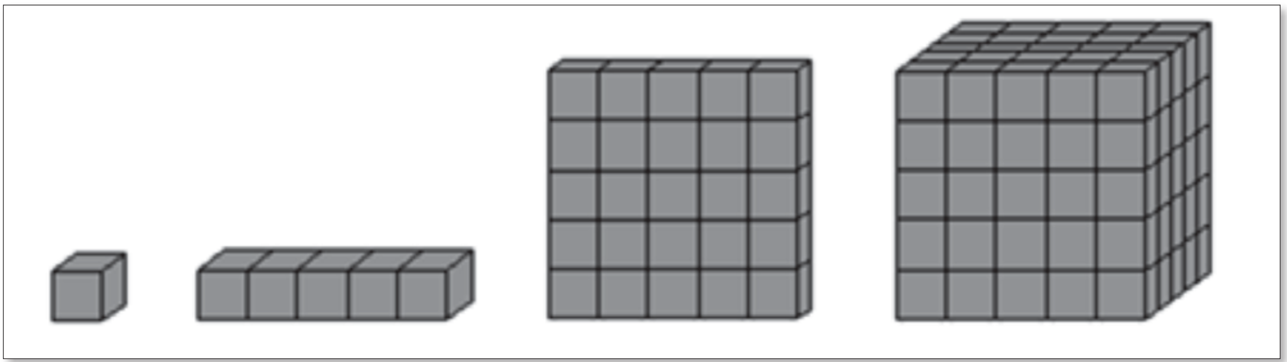

Figure 2. *Multibase Arithmetic Blocks* can be used to show squares and cubes for various bases.

This is the point from which the investigation described here proceeds, beginning with the single digit numbers. Students should develop the idea that repeated multiplication can be used to calculate larger index numbers, although since this process rapidly increases the size of the numbers involved, students should be encouraged to use digital technologies where appropriate.

Exploring powers of 5 and 6

In my experience, a brief interactive whiteboard demonstration is often sufficient for even very young students to observe that the powers of 6 (for example) all end in the digit 6: 6, 36, 216, 1296 etc. The SMART Board 'Powers' manipulative is capable of displaying the last digits of 6^1 through 6^{19} before the numbers become too large to display without scientific notation.

Similarly, the powers of 5 all end in 5: 5, 25, 125, 625, 3025 etc. The SMART Board manipulative allows all of the digits 5^1 through 5^{21} to be displayed. Students with keen observation skills may also notice that, with the exception of 5^1, the positive integral powers of 5 appear to share the last two digits—all of these index numbers end in 25.

Students may then be challenged to investigate whether there are any other single digit numbers for which this occurs. When such a search proves futile (easily demonstrated using the SMART Board manipulative), students can then be encouraged explore double digit numbers such as:

15 (15, 225, 3 375, 50 625 etc.),
25 (25, 625, 15 625, 390 625 etc.),
16 (16, 256, 4 096, 65 536, etc.), and
26 (26, 676, 17 576, 456 976 etc.).

Powers of 4 and 9

As students will readily discover, the last digit of 4^n alternates between 4 and 6 (i.e., a two digit cycle). A similar pattern is observed with the last digit of 9^n, which alternates between 9 and 1. At this point it is suggested that students begin to record the results of their investigation by completing a table such as the one shown in Figure 3.

Since the last digits of 4^n and 9^n repeat on a two digit cycle, students can be encouraged to refer to the pattern seen in the odd and even powers of these numbers. In other words, any odd power of 4 (i.e., 4^1, 4^3 etc.) ends in 4 while any even power (i.e., 4^2, 4^4 etc.) ends in 6. Similarly, the odd powers of 9 (i.e., 9^1, 9^3 etc.) end in 9 while the even powers (i.e., 9^2, 9^4 etc.) end in 1.

Powers of 2, 3, 7 and 8

Students should then be encouraged to explore the last digits of 2^n, 3^n, 7^n and 8^n, each of which repeats in a four digit cycle. By recording the results in a table, students should observe that, for example, the last digit of 2^3 is the same as that of 2^7. Some students may conjecture (correctly) that the pattern can be extended to 2^{11} and so on.

We can describe the pattern in the last digits of 2^n, 3^n, 7^n and 8^n in a succinct manner if we represent the last digit of 2^n as $LD\ (2^n)$. It should be acknowledged that few, if any, students at this level may attempt to describe their results using such a notation. Nevertheless, such attempts should be encouraged since students are expected to recognise and describe the patterns they observe.

Since the last digits of 2^n, 3^n, 7^n and 8^n repeat on a four digit cycle, this pattern can be expressed quite succinctly as $LD\ (2^n) = LD\ (2^{n+4})$. By recognising and describing this pattern, the sequence

		Power								
		1	2	3	4	5	6	7	8	9
Base	1	1	1	1	1	1	1	1	1	1
	2	2	4	8	6	2	4	8	6	2
	3	3	9	7	1	3	9	7	1	3
	4	4	6	4	6	4	6	4	6	4
	5	5	5	5	5	5	5	5	5	5
	6	6	6	6	6	6	6	6	6	6
	7	7	9	3	1	7	9	3	1	7
	8	8	4	2	6	8	4	2	6	8
	9	9	1	9	1	9	1	9	1	9
	10	0	0	0	0	0	0	0	0	0

Figure 3. The last digits of the numbers 1–10 raised to various powers repeat in 1, 2 or 4 digit cycles.

can be extended to calculate, for example, $LD\ (2^{100})$ or even $LD\ (2^{1000})$. Recognising that $LD\ (3^n) = LD\ (3^{n+4})$ allows us to greatly simplify the original problem by subtracting multiples of 4 from the power. That is, $LD\ (3^{2015}) = LD\ (3^{15})$, since 2000 is a multiple of 4. Continuing in this manner, we see that $LD\ (3^{15}) = LD\ (3^{11}) = LD\ (3^7) = LD\ (3^3)$, which is 7. Many students find this idea highly motivating, since such calculations are generally beyond the ability of most calculators to display.

Powers of 0 and 1

The multiplication property of zero means that the last digit of 0^n will always be 0. Similarly, the multiplication property of one means that the last digit of 1^n will always be 1. While these results may seem trivial in themselves, students should be encouraged to explore the last digits of 10^n, since a clear understanding of the pattern in the powers of 10 underpins the concept of whole number and decimal place value.

By extending their investigation to the last digits of 11^n, 12^n and beyond (see Figure 4), students should discover that the last digits of the numbers 11–20 raised to various powers follows exactly the same pattern as that for the numbers 1–10. Students who require further convincing can then explore the last digits of larger index numbers such as 20^n and 21^n. This exploration has the potential to generate some interesting discussion about the nature of mathematical proof. Students can be prompted with questions such as "How can you be certain that the pattern will continue?"and "How long do you think the pattern will continue?" Students should be encouraged to develop their own conjectures, such as that the last digit of an index number is independent of the digit in the tens (and later, hundreds) place and instead it depends only on the digit in the units place.

The proof of this conjecture is provided here as a potentially useful piece of teacher knowledge. That is, when two integers are multiplied together, the last digit of the product depends only on the last digits of the original integers.

		Power								
		1	2	3	4	5	6	7	8	9
Base	1	1	1	1	1	1	1	1	1	1
	2	2	4	8	6	2	4	8	6	2
	3	3	9	7	1	3	9	7	1	3
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	11	1	1	1	1	1	1	1	1	1
	12	2	4	8	6	2	4	8	6	2
	13	3	9	7	1	3	9	7	1	3

Figure 4. The last digits of the numbers 11–20 raised to various powers follow the same pattern as for 1–10.

This can be illustrated by expressing the first integer as $10a + b$ and multiplying by any other integer $(10c + d)$. The product of these expressions, $100ac + 10ad + 10bc + bd$ ($= 10k + bd$) reveals that when looking at final digits the only relevant product is the product of the final digits of the original integers. While proving such a conjecture is acknowledged as well beyond most students at this level, the fact that $10a + b$ represents a general two-digit integer may be a useful teaching point. This is because many students' first attempt to represent a general two-digit integer is '*ab*', revealing the need to further consolidate their understanding of place value.

Conclusion

The investigation described here emphasises the usefulness of looking for patterns and solving a simpler problem, as problem-solving techniques. The activity is suggested as a means for teachers to extend students' understanding of index notation, powers and place value. Such exploration builds number sense by allowing students to become more comfortable and familiar with number meanings and operations and to use them in flexible ways (Anghileri, 2006).

While the activity described here restricts its attention to the last digits of counting numbers raised to positive integer powers, the activity can of course be modified to suit student needs. For example, the activity could readily lead on to the exploration of digit problems such as finding those two-digit numbers (e.g., 36) which are equal to twice the product of their digits (i.e., those values of a and b for which $10a + b = 2ab$). Conversely, exploring the last digits in the 1×1 through 9×9 multiplication tables represents a much more accessible investigation. Swan (2007) suggests this remains fertile ground for student exploration.

References

Anghileri, J. (2006). *Teaching numbers sense* (2nd ed.). London: Continuum.

Atkins, W. J., & Taylor, P. J. (2012). *Australian mathematics competition book 5, 2006–2012.* Canberra: Australian Mathematics Trust.

Australian Curriculum and Reporting Authority. (2014). *The Australian Curriculum: Mathematics.* http://www.australiancurriculum.edu.au/

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2014). *Teaching primary mathematics* (5th ed.). NSW: Pearson Australia.

Haylock, D., & Manning, R. (2014). *Mathematics explained for primary teachers* (5th ed.). London: SAGE Publications.

Posamentier, A., & Krulik, S. (2012). T*he art of motivating students for mathematics instruction.* NY: McGraw-Hill.

Swan, P. (2007). *Tackling tables: Using a strategies approach* (2nd ed.). Perth, Western Australia: A–Z Type.